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Resolvent Estimates in L^p for Elliptic Systems in Lipschitz Domains

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We obtain the L^p resolvent estimates in Lipschitz domains in $\mathbb{R}^n$ for constant coefficient elliptic systems satisfying the Legendre–Hadamard condition, where $1 \leqslant p \leqslant \infty$ for $n=3$; and $2n/(n+3)-\delta < p < 2n/(n-3)+\delta$ for $n \geqslant 4$ and some $\delta = \delta(\Omega) > 0$.

0. INTRODUCTION

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$ with connected boundary. Consider the boundary value problem for the elliptic system:

$$
\begin{aligned}
-a_{ij}^{\alpha\beta} \frac{\partial^2 u^\beta}{\partial x_i \partial x_j} + \lambda u^\alpha = f^\alpha \qquad & \text{in } \Omega, \quad \alpha = 1, 2, ..., m \\
\mathbf{u} = \mathbf{0} \qquad & \text{on } \partial\Omega
\end{aligned} \tag{0.1}
$$

where $\lambda \in \mathbb{C} \setminus (-\infty, 0)$ is a complex parameter. We will assume that $a_{ij}^{\alpha\beta}$, $1 \leqslant i, j \leqslant n$, $1 \leqslant \alpha$, $\beta \leqslant m$ are real constants with symmetry property $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$ and satisfy the Legendre–Hadamard condition:

$$
\mu\, |\xi|^2\, |\eta|^2 \leqslant a_{ij}^{\alpha\beta} \xi_i\, \xi_j\, \eta^\alpha \eta^\beta \leqslant \frac{1}{\mu}\, |\xi|^2\, |\eta|^2 \tag{0.2}
$$

for some $\mu > 0$ and any $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$.

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It is well known that, if $\mathbf{f} \in L^2(\Omega)$, then (0, 1) has a unique solution in $H_0^1(\Omega)$. Moreover, by the energy estimate,

$$\|\mathbf{u}\|_{L^2(\Omega)} \leqslant \frac{C}{(1+|\lambda|)} \|\mathbf{f}\|_{L^2(\Omega)},$$

$$\|\nabla \mathbf{u}\|_{L^2(\Omega)} \leqslant \frac{C}{(1+|\lambda|)^{1/2}} \|\mathbf{f}\|_{L^2(\Omega)}.$$

In this paper, we shall be interested in the similar estimate in L^p spaces.

For $\theta \in (0, \pi/2)$, let $\Sigma_\theta = \{\lambda \in \mathbb{C} : \lambda = 0 \text{ or } |\arg \lambda| < \pi - \theta\}$. The main results of this paper are as follows.

THEOREM 0.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$ with connected boundary and $\lambda \in \Sigma_\theta$ for some $\theta > 0$.*

(a) *If $n \geqslant 4$, there exist constants $C > 0$, $\delta > 0$ depending on Ω, n, θ and the ellipticity constant μ, such that, if $\mathbf{f} \in L^p(\Omega) \cap L^2(\Omega)$, $2n/(n+3) - \delta < p < 2n/(n-3) + \delta$, then the unique solution $\mathbf{u}$ of (0.1) in $H_0^1(\Omega)$ satisfies*

$$\|\mathbf{u}\|_{L^p(\Omega)} \leqslant \frac{C}{1+|\lambda|} \|\mathbf{f}\|_{L^p(\Omega)}, \tag{0.4}$$

(b) *If $n = 3$, (0.4) holds for $1 \leqslant p \leqslant \infty$.*

THEOREM 0.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$ with connected boundary and $\lambda \in \Sigma_\theta$ for some $\theta > 0$. There exist constants $C > 0$, $\delta > 0$ depending on Ω, n, θ and μ, such that, if $\mathbf{f} \in L^p(\Omega) \cap L^2(\Omega)$, $2n/(n+1) - \delta < p < 2n/(n-1) + \delta$, then the unique solution $\mathbf{u}$ of (0.1) in $H_0^1(\Omega)$ satisfies*

$$\|\nabla \mathbf{u}\|_{L^p(\Omega)} \leqslant \frac{C}{(1+|\lambda|)^{1/2}} \|\mathbf{f}\|_{L^p(\Omega)}. \tag{0.6}$$

If we let A denote the elliptic operator given by

$$(A\mathbf{u})^\alpha = a_{ij}^{\alpha\beta} \frac{\partial^2 u^\beta}{\partial x_i \partial x_j}, \tag{0.7}$$

then (0.4) implies that

$$\|(-A+\lambda I)^{-1} \mathbf{f}\|_{L^p(\Omega)} \leqslant \frac{C}{1+|\lambda|} \|\mathbf{f}\|_{L^p(\Omega)} \tag{0.8}$$

for $\lambda \in \Sigma_\theta$, $\theta > 0$. Consequently, $-A$ generates a bounded analytic semigroup in $L^p(\Omega)$ ($n=3$, $1<p<\infty$; $n\geqslant 4$, $2n/(n+3)-\delta<p<2n/(n-3)+\delta$).

To prove Theorem 0.3 and 0.5, we need to study the Dirichlet problem with boundary data in $L^p(\partial\Omega)$:

$$\begin{aligned} -A\mathbf{u}+\lambda\mathbf{u}&=\mathbf{0} \quad \text{in } \Omega\\ \mathbf{u}&=\mathbf{g}\in L^p(\partial\Omega) \quad \text{on } \partial\Omega\\ (\mathbf{u})^* &\in L^p(\partial\Omega) \end{aligned} \tag{0.9}$$

where $(\mathbf{u})^*$ denotes the nontangential maximal function of $\mathbf{u}$.

THEOREM 0.10. *Let Ω be a bounded Lipschitz domain with connected boundary in $\mathbb{R}^n$, $n\geqslant 3$ and $\lambda\in\Sigma_\theta$ for some $\theta>0$. There exists $\delta>0$ depending on the Lipschitz constant of Ω, n, θ and μ, such that, if $2-\delta<p<2+\delta$, $\mathbf{g}\in L^p(\partial\Omega)$, then* (0.9) *has a unique solution. Moreover, the solution will satisfy*

$$\|(\mathbf{u})^*\|_{L^p(\partial\Omega)} \leqslant C\,\|\mathbf{g}\|_{L^p(\partial\Omega)} \tag{0.11}$$

where C depends only on Ω, n, θ and μ.

THEOREM 0.12. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n\geqslant 3$ with connected boundary. If $\mathbf{g}\in L^2_1(\partial\Omega)$, i.e., $\mathbf{g}$ has the first-order derivatives in $L^2(\partial\Omega)$, then there exists a unique solution $\mathbf{u}$ satsfying* (0.9) *and $(\nabla\mathbf{u})^*\in L^2(\partial\Omega)$. In fact, the solution will satisfy*

$$\|(\nabla\mathbf{u})^*\|_{L^2(\partial\Omega)} \leqslant C\{\|\nabla_{\tan}\mathbf{g}\|_{L^2(\partial\Omega)} + \|\,|\lambda|^{1/2}\mathbf{g}\|_{L^2(\partial\Omega)} + \|\mathbf{g}\|_{L^2(\partial\Omega)}\} \tag{0.13}$$

where C depends only on Ω, n, θ and μ.

In Theorem 0.12, $\nabla_{\tan}\mathbf{g}$ denotes the tangential derivatives of $\mathbf{g}$ on $\partial\Omega$.

We remark that, for the system of elastostatics,

$$\mu\,\Delta\mathbf{u}+(\lambda+\mu)\,\nabla(\operatorname{div}\mathbf{u})=\mathbf{0}$$

in Ω, the solvability of boundary value problems with data in L^p for $2-\delta<p<2+\delta$ was established in [7] by the method of layer potentials. The key step is to show that, for solutions of the system of elastostatics, the L^2 norm of the conormal derivative on the boundary is equivalent to the L^2 norm of its tangential derivatives. For the general elliptic systems satisfying the Legendre–Hadamard condition (0.2), such Rellich type equivalence between the conormal derivative and the tangential derivatives fails. Nevertheless, for the case $\lambda=0$, Theorem 0.10 was proved by W. Gao in his thesis [9], based on an idea of A. P. Calderón [3]. Mimicking the argument in [3, 9], one may produce the existence and uniqueness of the solution of (0.9) for $\lambda \in \Sigma_\theta$. However, to establish estimate (0.11) with constant C *independent* of λ, which is essential in this paper, we need to appeal to the technique developed recently by J. Pipher and G.Verchota [15]. The basic idea is to represent the solution by the Green's formula in terms of its Cauchy data and use the Rellich type inequalities to estimate the term with the higher order data. For the latter, we will prove that, in the region D above a Lipschitz graph, for a solution of $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0}$ in D,

$$\|\nabla_{\tan}\mathbf{u}\|_{L^2(\partial D)} + \| |\lambda|^{1/2}\mathbf{u}\|_{L^2(\partial D)} \sim \left\|\frac{\partial \mathbf{u}}{\partial x_n}\right\|_{L^2(\partial D)} \tag{0.14}$$

(see Lemma 1.19). Note that in (0.14) we use the derivative of $\mathbf{u}$ in the direction $\mathbf{e}_n$ in the place usually occupied by the conormal derivatives.

Using estimate (0.11) and tehniques in [5, 14, 15], we are able to estimate the L^1 norm of the Green's function $G_\lambda(X, Y)$ in Ω. We show that, in the case $n=3$,

$$\int_\Omega |G_\lambda(X, Y)|\, dY \leqslant \frac{C}{1+|\lambda|} \tag{0.15}$$

and for $n \geqslant 4$,

$$\int_{\Omega\setminus B(X, r/2)} |G_\lambda(X, Y)|\, dY \leqslant \frac{C}{r^{(n-3)/2-\varepsilon}(1+|\lambda|)^{(n+1)/4-(\varepsilon/2)}} \tag{0.16}$$

where $\varepsilon=\varepsilon(\Omega)>0$ and $r=\operatorname{dist}(X, \partial\Omega)$ (see Lemma 3.4). Clearly, (0.15) implies (0.4) for $n=3$, $1\leqslant p\leqslant\infty$. For $n\geqslant 4$, (0.16) gives certain weak L^∞-estimates. This, together with the appropriate L^2-estimates, yields Theorem 0.3 by interpolation and duality for the case of $n\geqslant 4$. Theorem 0.5 follows in the same manner.

We note that, for smooth domains, the resolvent estimate (0.4) follows for the a-priori estimates in $W^{2,p}(\Omega)$, together with an argument of S. Agmon. It is known that such a-priori estimate in $W^{2,p}$ fails in general for Lipschitz domains. However, for the single equation ($m=1$), estimate (0.4) can be proved by a rather simple argument for $1\leqslant p\leqslant\infty$ in any dimension (see [13, p. 215]). For general systems in Lipschitz domains, we do not know whether the restriction of p for $n\geqslant 4$ is necessary. Clearly, this is closely related to a more basic problem: the weak maximum principle

$$\|\mathbf{u}\|_{L^\infty(\Omega)} \leqslant C\, \|\mathbf{u}\|_{L^\infty(\partial\Omega)} \tag{0.17}$$

for solutions of $\mathcal{A}\mathbf{u}=\mathbf{0}$ in the Lipschitz domain Ω. For $n=3$, the L^∞ estimate (0.17) has been established by B. Dahlberg and C. Kenig [5]. Also see [14, 15] for higher order equations and [18] for the Stokes system. The problem remains open for $n \geqslant 4$.

The outline of this paper is as follows. In Section 1, we derive some Rellich type inequalities. Section 2 is devoted to the proof of Theorem 0.10 and 0.12. Our main results, Theorem 0.3 and 0.5, are proved in Section 3.

Throughout the paper we will assume $\lambda \in \Sigma_\theta$ for some $\theta > 0$. We will use C and c to denote positive constants depending at most on Ω, n, θ and μ.

1. Rellich Estimates

In this section we derive some Rellich type estimates for the system $-\mathcal{A}\mathbf{u}+\lambda\mathbf{u}=\mathbf{f}$ in Ω where $\lambda \in \Sigma_\theta$. Similar etimates have been obtained in [1, 2, 7–12, 15, 16, 20] for various equations and systems.

We begin with Rellich–Payne–Weinberger–Nečas identities. Using symmetry condition $a_{ij}^{\alpha\beta}=a_{ji}^{\beta\alpha}$, a computation shows

$$\frac{\partial}{\partial x_l}\left[(h_l a_{ij}^{\alpha\beta}-h_i a_{lj}^{\alpha\beta}-h_j a_{il}^{\alpha\beta})\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j}\right]$$
$$=-2Re\left[h_i\frac{\partial \bar{u}^\alpha}{\partial x_i}(\mathcal{A}\mathbf{u})^\alpha\right]+\operatorname{div}\mathbf{h}\cdot a_{ij}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j}$$
$$-2Re\left[\frac{\partial h_i}{\partial x_l}a_{lj}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j}\right]$$

where $\bar{u}^\alpha$ denotes the complex conjugate of u^α and Re the real part. It then follows from the divergence theorem that

$$\int_{\partial\Omega}\langle\mathbf{h},\mathbf{N}\rangle\, a_{ij}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j}=2Re\int_{\partial\Omega}N_l h_i a_{lj}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j}$$
$$-2Re\int_\Omega h_i\frac{\partial \bar{u}^\alpha}{\partial x_i}(\mathcal{A}\mathbf{u})^\alpha+\int_\Omega \operatorname{div}\mathbf{h}\cdot a_{ij}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j}$$
$$-2Re\int_\Omega\frac{\partial h_i}{\partial x_l}\cdot a_{lj}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j} \tag{1.1}$$

where $\mathbf{N}=(N_1, N_2, ..., N_n)$ denotes the outward unit normal to $\partial\Omega$. Subtracting each side of (1.1) from double the left side of (1.1), we obtain

$$\int_{\partial\Omega} \langle \mathbf{h}, \mathbf{N} \rangle\, a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial \bar{u}^\beta}{\partial x_j} = 2Re \int_{\partial\Omega} (h_l N_l a_{ij}^{\alpha\beta} - h_i N_l a_{lj}^{\alpha\beta}) \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial \bar{u}^\beta}{\partial x_j}$$
$$+ 2Re \int_\Omega h_i \frac{\partial \bar{u}^\alpha}{\partial x_i} (A\mathbf{u})^\alpha - \int_\Omega \operatorname{div} \mathbf{h} \cdot a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial \bar{u}^\beta}{\partial x_j}$$
$$+ 2Re \int_\Omega \frac{\partial h_i}{\partial x_l} \cdot a_{lj}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial \bar{u}^\beta}{\partial x_j}.$$

LEMMA 1.3. *Let* $\mathbf{u} \in C^2(\bar{\Omega})$. *Suppose that*

$$\begin{cases} -A\mathbf{u} + \lambda \mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega. \end{cases}$$

Then

$$(1+|\lambda|)^{1/2} \int_\Omega |\mathbf{u}|^2 \leqslant C \int_{\partial\Omega} |\mathbf{g}|^2 + \frac{C}{(1+|\lambda|)^{3/2}} \int_\Omega |\mathbf{f}|^2.$$

Proof. Let $\mathbf{F} \in C_0^\infty(\Omega)$ and $\mathbf{v}$ solve the boundary value problem:

$$\begin{cases} -A\mathbf{v} + \bar{\lambda} \mathbf{v} = \mathbf{F} \\ \mathbf{v} \in H_0^1(\Omega). \end{cases} \tag{1.4}$$

It follows from integration by parts that

$$\int_\Omega \langle \mathbf{u}, \mathbf{F} \rangle = \int_\Omega \langle \mathbf{f}, \mathbf{v} \rangle - \int_{\partial\Omega} \left\langle \mathbf{u}, \frac{\partial \mathbf{v}}{\partial \nu} \right\rangle$$

where $\partial \mathbf{v}/\partial \nu$ denotes the conormal derivative given by

$$\left(\frac{\partial \mathbf{v}}{\partial \nu}\right)^\alpha = a_{ij}^{\alpha\beta} \frac{\partial v^\beta}{\partial x_j} N_i. \tag{1.5}$$

Thus, by the Schwartz inequality,

$$\left| \int_\Omega \langle \mathbf{u}, \mathbf{F} \rangle \right| \leqslant \left(\int_\Omega |\mathbf{f}|^2 \right)^{1/2} \left(\int_\Omega |\mathbf{v}|^2 \right)^{1/2} + \left(\int_{\partial\Omega} |\mathbf{g}|^2 \right)^{1/2} \left(\int_{\partial\Omega} \left| \frac{\partial \mathbf{v}}{\partial \nu} \right|^2 \right)^{1/2}.$$

We claim that

$$\int_\Omega |\nabla \mathbf{v}|^2 + (1+|\lambda|) \int_\Omega |\mathbf{v}|^2 \leqslant \frac{C}{1+|\lambda|} \int_\Omega |\mathbf{F}|^2, \tag{1.6}$$

$$\int_{\partial\Omega} |\nabla \mathbf{v}|^2 \leqslant \frac{C}{(1+|\lambda|)^{1/2}} \int_\Omega |\mathbf{F}|^2. \tag{1.7}$$

Lemma 1.3 follows easily from (1.6) and (1.7) by duality.

To see (1.6) we have, by (1.4),

$$\int_\Omega a_{ij}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_i} \cdot \frac{\partial \bar{v}^\beta}{\partial x_j} + \lambda \int_\Omega |\mathbf{v}|^2 = \int_\Omega \langle \mathbf{v}, \mathbf{F} \rangle. \tag{1.8}$$

Since $\mathbf{v} \in H_0^1(\Omega)$,

$$c \int_\Omega |\nabla \mathbf{v}|^2 \leqslant \int_\Omega a_{ij}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_i} \cdot \frac{\partial \bar{v}^\beta}{\partial x_j}.$$

Also, note that, if $\lambda \in \Sigma_\theta$ and $Re\lambda < 0$, then $|\lambda| \leqslant C_\theta |Im\lambda|$. By taking the real part and imaginary part of (1.8), it follows that, for any $\lambda \in \Sigma_\theta$,

$$\int_\Omega |\nabla \mathbf{v}|^2 + |\lambda| \int_\Omega |\mathbf{v}|^2 \leqslant C \left(\int_\Omega |\mathbf{F}|^2 \right)^{1/2} \left(\int_\Omega |\mathbf{v}|^2 \right)^{1/2}. \tag{1.9}$$

Thus

$$\int_\Omega |\nabla \mathbf{v}|^2 + |\lambda| \int_\Omega |\mathbf{v}|^2 \leqslant \frac{C}{|\lambda|} \int_\Omega |\mathbf{F}|^2.$$

This proves (1.6) for $|\lambda| \geqslant 1$. If $|\lambda| \leqslant 1$, (1.6) follows easily from (1.9) and the Poincaré inequality.

We now turn to the proof of (1.7). Choose $\mathbf{h} \in C_0^\infty(\mathbb{R}^n)$ such that $\langle \mathbf{h}, \mathbf{N} \rangle \geqslant c_0 > 0$ on $\partial\Omega$. Note that, for each α, β, j fixed, the vector $h_l N_l a_{ij}^{\alpha\beta} - h_i N_l a_{lj}^{\alpha\beta}$ is perpendicular to $\mathbf{N}$. Hence,

$$\left| (h_l N_l a_{ij}^{\alpha\beta} - h_i N_l a_{lj}^{\alpha\beta}) \frac{\partial v^\alpha}{\partial x_i} \right| \leqslant C |\nabla_{\tan} \mathbf{v}| = 0 \qquad \text{on } \partial\Omega.$$

It then follows from (1.2) that

$$\int_{\partial\Omega} \langle \mathbf{h}, \mathbf{N} \rangle a_{ij}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_i} \cdot \frac{\partial \bar{v}^\beta}{\partial x_j} \leqslant C \int_\Omega |\nabla \mathbf{v}|^2 + C \int_\Omega |\nabla \mathbf{v}|\, |\lambda \mathbf{v}| + C \int_\Omega |\nabla \mathbf{v}|\, |\mathbf{F}|. \tag{1.10}$$

It is known that

$$c\,|\nabla \mathbf{v}|^2 \leqslant a_{ij}^{\alpha\beta}\frac{\partial v^\alpha}{\partial x_i}\cdot\frac{\partial \mathbf{v}^\beta}{\partial x_j}+C\,|\nabla_{\tan}\mathbf{v}|^2 \qquad \text{on } \partial\Omega \tag{1.11}$$

(e.g. see [11, p. 168]). Thus, by (1.10), (1.11), we have

$$c\int_{\partial\Omega}|\nabla \mathbf{v}|^2 \leqslant C\int_\Omega |\nabla \mathbf{v}|^2 + C\int_\Omega |\nabla \mathbf{v}|\,|\lambda \mathbf{v}| + C\int_\Omega |\nabla \mathbf{v}|\,|\mathbf{F}|.$$

(1.7) now follows from the Schwartz inequality and (1.6). The proof is complete.

Remark 1.12. The same argument as in the proof of Lemma 1.3 also shows that, if $D=\{(X', x_n)\in\mathbb{R}^n,\ x_n>\varphi(X')\}$ is a region above a Lipschitz graph and $\mathbf{u}\in C^2(\bar{D})$ (suitably small at ∞) solves

$$-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0} \qquad \text{in } D$$
$$\mathbf{u}=\mathbf{g} \qquad \text{on } \partial D,$$

then

$$|\lambda|^{1/2}\int_D |\mathbf{u}|^2 \leqslant C\int_{\partial D}|\mathbf{g}|^2.$$

We will use $\|\mathbf{u}\|_p$ to denote the norm of $\mathbf{u}$ in $L^p(\Omega)$ and $\|\mathbf{u}\|_\partial$ the norm of $\mathbf{u}$ in $L^2(\partial\Omega)$.

LEMMA 1.13. *Suppose* $\mathbf{u}\in C^2(\bar{\Omega})$ *and* $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0}$ *in* Ω. *Then*

(a) $\|\nabla\mathbf{u}\|_2^2 \leqslant C\{\|\mathbf{u}\|_\partial^2+\|\nabla\mathbf{u}\|_\partial\,\|\mathbf{u}\|_\partial+\|\,|\lambda|^{1/4}\mathbf{u}\|_\partial^2\}$,

(b) $\|\,|\lambda|^{1/4}\,\nabla\mathbf{u}\|_2^2+\|\,|\lambda|^{3/4}\mathbf{u}\|_2^2 \leqslant C\{\|\nabla\mathbf{u}\|_\partial\,\|\,|\lambda|^{1/2}\mathbf{u}\|_\partial+\|\,|\lambda|^{1/2}\,\mathbf{u}\|_\partial^2+\|\mathbf{u}\|_\partial^2\}$.

Proof. We start with part (a).

Let $\mathbf{g}=\mathbf{u}|_{\partial\Omega}$ and $\mathbf{v}$ be the harmonic extension of $\mathbf{g}$ into Ω. Then $\mathbf{u}-\mathbf{v}\in H_0^1(\Omega)$. It follows that

$$\begin{aligned}
c\int_\Omega |\nabla(\mathbf{u}-\mathbf{v})|^2 &\leqslant \int_\Omega a_{ij}^{\alpha\beta}\frac{\partial}{\partial x_i}(u^\alpha-v^\alpha)\frac{\partial}{\partial x_j}(\bar{u}^\beta-\bar{v}^\beta)\\
&\leqslant \int_\Omega a_{ij}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j}+C\int_\Omega |\nabla\mathbf{u}|\,|\nabla\mathbf{v}|+C\int_\Omega|\nabla\mathbf{v}|^2\\
&\leqslant C\{\|\nabla\mathbf{u}\|_\partial\,\|\mathbf{u}\|_\partial+|\lambda|\,\|\mathbf{u}\|_2^2+\|\nabla\mathbf{u}\|_2\,\|\nabla\mathbf{v}\|_2+\|\nabla\mathbf{v}\|_2^2\}.
\end{aligned}$$

By Lemma 1.3, $|\lambda|\ \|\mathbf{u}\|_2^2 \leqslant C\ \|\,|\lambda|^{1/4}\mathbf{u}\|_\partial^2$. Hence,

$$\|\nabla\mathbf{u}\|_2^2 \leqslant C\{\|\nabla\mathbf{u}\|_\partial\ \|\mathbf{u}\|_\partial + \|\,|\lambda|^{1/4}\,\mathbf{u}\|_\partial^2 + \|\nabla\mathbf{v}\|_2^2\}$$

By the estimate of harmonic functions in Ω (e.g. see [20]),

$$\begin{aligned}\|\nabla\mathbf{v}\|_2^2 &\leqslant \|\nabla\mathbf{v}\|_\partial\ \|\mathbf{v}\|_\partial \leqslant C\ \|\mathbf{g}\|_{L_1^2(\partial\Omega)}\ \|\mathbf{u}\|_\partial \\ &\leqslant C\ \|\nabla\mathbf{u}\|_\partial\ \|\mathbf{u}\|_\partial + C\ \|\mathbf{u}\|_\partial^2.\end{aligned}$$

Part (a) then follows.

Part (b) follows from Part (a) and Lemma 1.3.

LEMMA 1.14. *With the same hypotheses as Lemma* 1.13 *we have*

$$\|\nabla\mathbf{u}\|_\partial \leqslant C\{\|\nabla_{\tan}\mathbf{u}\|_\partial + \|\,|\lambda|^{1/2}\,\mathbf{u}\|_\partial + \|\mathbf{u}\|_\partial\}.$$

Proof. It follows from (1.11), (1.2) and the Schwartz inequality that

$$\begin{aligned}\|\nabla\mathbf{u}\|_\partial^2 \leqslant{}& C\{\|\nabla_{\tan}\mathbf{u}\|_\partial^2 + \|\nabla_{\tan}\mathbf{u}\|_\partial\ \|\nabla\mathbf{u}\|_\partial \\ &+ \|\,|\lambda|^{1/4}\,\nabla\mathbf{u}\|_2\ \|\,|\lambda|^{3/4}\mathbf{u}\|_2 + \|\nabla\mathbf{u}\|_2^2\} \\ \leqslant{}& C\{\|\nabla_{\tan}\mathbf{u}\|_\partial^2 + \|\nabla\mathbf{u}\|_\partial + \|\nabla\mathbf{u}\|_\partial\ \|\,|\lambda|^{1/2}\,\mathbf{u}\|_\partial \\ &+ \|\,|\lambda|^{1/2}\mathbf{u}\|_\partial^2 + \|\mathbf{u}\|_\partial^2 + \|\nabla\mathbf{u}\|_\partial\ \|\mathbf{u}\|_\partial\}\end{aligned}$$

where in the second inequality we have used Lemma 1.13. Lemma 1.14 then follows from a usual trick.

Remark 1.15. Lemma 1.14 also holds in $\bar{\Omega}^c$ if, in addition, we assume that $\mathbf{u}$ and its derivatives are suitably small at ∞.

In the rest of this section we will establish a Rellich type estimate on the region D above a Lipschitz graph:

$$D = \{(X', x_n) \in \mathbb{R}^n : x_n > \varphi(X'),\ X' \in \mathbb{R}^{n-1}\}$$

where φ is a Lipschitz function, i.e., $|\varphi(X') - \varphi(Y')| \leqslant M\,|X' - Y'|$.

LEMMA 1.16. *Let* $\mathbf{u} \in C^1(\bar{D})$. *Assume that* $|\nabla\mathbf{u}|$ *is suitably small at* ∞. *Then*

$$c\int_{\partial D} |\nabla\mathbf{u}|^2 \leqslant \int_{\partial D} \langle -\mathbf{e}_n, \mathbf{N}\rangle\, a_{ij}^{\alpha\beta}\,\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial \bar{u}^\beta}{\partial x_j} + C\int_{\partial D}\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|^2$$

where $\mathbf{e}_n = (0, 0, ..., 1)$.

Proof. Let $\mathbf{v}(X') = \mathbf{u}(X', \varphi(X'))$ for $X' \in \mathbb{R}^{n-1}$. Note that

$$\frac{\partial \mathbf{v}}{\partial x_l}(X') = \frac{\partial \mathbf{u}}{\partial x_l}(X', \varphi(X')) + \frac{\partial \mathbf{u}}{\partial x_n}(X', \varphi(X')) \cdot \frac{\partial \varphi}{\partial x_l}(X') \tag{1.17}$$

for $l = 1, 2, ..., n-1$. Thus,

$$c \int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \leqslant c \int_{\mathbb{R}^{n-1}} \left| \frac{\partial \mathbf{v}}{\partial X'} \right|^2 dX' + C \int_{\partial D} \left| \frac{\partial \mathbf{u}}{\partial x_n} \right|^2 d\sigma$$

$$\leqslant \int_{\mathbb{R}^{n-1}} a_{lt}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_l} \cdot \frac{\partial \bar{v}^\beta}{\partial x_t} dX' + C \int_{\partial D} \left| \frac{\partial \mathbf{u}}{\partial x_n} \right|^2 d\sigma$$

where $1 \leqslant l, t \leqslant n-1$ and we have used the Fourier transform on $\mathbb{R}^{n-1}$ for the second inequality.

Now, by (1.17),

$$a_{lt}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_l}(X') \cdot \frac{\partial \bar{v}^\beta}{\partial x_t}(X') \leqslant a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}(X', \varphi(X')) \cdot \frac{\partial \bar{u}^\beta}{\partial x_j}(X', \varphi(X'))$$

$$+ C\, |\nabla \mathbf{u}(X', \varphi(X'))| \left| \frac{\partial \mathbf{u}}{\partial x_n}(X', \varphi(X')) \right|$$

where $1 \leqslant l, t \leqslant n-1$ and $1 \leqslant i, j \leqslant n$. Also, note that the outward unit normal $\mathbf{N} = (\partial\varphi/\partial x_1, ..., \partial\varphi/\partial x_{n-1}, -1)/\sqrt{1 + |\nabla\varphi|^2}$ and $d\sigma$, the surface measure on ∂D, $= \sqrt{1 + |\nabla\varphi|^2}\, dX'$. Thus,

$$\langle -\mathbf{e}_n, \mathbf{N} \rangle \, d\sigma = dX'. \tag{1.18}$$

It follows that

$$c \int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \leqslant \int_{\mathbb{R}^{n-1}} a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}(X', \varphi(X')) \frac{\partial \bar{u}^\beta}{\partial x_j}(X', \varphi(X'))\, dX'$$

$$+ C \int_{\partial D} |\nabla \mathbf{u}| \left| \frac{\partial \mathbf{u}}{\partial x_n} \right| d\sigma + C \int_{\partial D} \left| \frac{\partial \mathbf{u}}{\partial x_n} \right|^2 d\sigma$$

$$= \int_{\partial D} \langle -\mathbf{e}_n, \mathbf{N} \rangle \, a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial \bar{u}^\beta}{\partial x_j} d\sigma$$

$$+ C \int_{\partial D} |\nabla \mathbf{u}| \left| \frac{\partial \mathbf{u}}{\partial x_n} \right| d\sigma + C \int_{\partial D} \left| \frac{\partial \mathbf{u}}{\partial x_n} \right|^2 d\sigma.$$

The lemma then follows easily.

We now state the Rellich estimate on D.

LEMMA 1.19. *Suppose* $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0}$ *in* D *and* $\mathbf{u}$ *and its derivatives are suitably small at* ∞. *Then*

$$\|\nabla\mathbf{u}\|_{L^2(\partial D)}+\| |\lambda|^{1/2}\mathbf{u}\|_{L^2(\partial D)}\leqslant C\left\|\frac{\partial\mathbf{u}}{\partial x_n}\right\|_{L^2(\partial D)}.$$

Proof. Letting $\mathbf{h}=-\mathbf{e}_n=(0,0,...,-1)$ in (1.1), we obtain

$$\int_{\partial D}\langle -\mathbf{e}_n,\mathbf{N}\rangle\, a_{ij}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial\bar{u}^\beta}{\partial x_j}\leqslant C\int_{\partial D}|\nabla\mathbf{u}|\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|+C\int_D\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|\cdot|\lambda|\,|\mathbf{u}|.$$

By Lemma 1.16 and the Schwartz inequality

$$\int_{\partial D}|\nabla\mathbf{u}|^2\leqslant C\int_{\partial D}\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|^2+C\int_D\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|\,|\lambda|\,|\mathbf{u}|.$$

Also, note that

$$\begin{aligned}c\int_{\partial D}|\mathbf{u}|^2\leqslant\int_{\partial D}\langle -\mathbf{e}_n,\mathbf{N}\rangle\,|\mathbf{u}|^2&=-2Re\int_D\left\langle\mathbf{u},\frac{\partial\mathbf{u}}{\partial x_n}\right\rangle\\&\leqslant 2\int_D|\mathbf{u}|\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|.\end{aligned}$$

It follows that

$$\int_{\partial D}|\nabla\mathbf{u}|^2+\int_{\partial D}|\lambda|\,|\mathbf{u}|^2\leqslant C\int_{\partial D}\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|^2+C\int_D\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|\,|\lambda|\,|\mathbf{u}|.\tag{1.20}$$

By Remark 1.12,

$$\int_D|\lambda|^{1/2}\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|^2\leqslant C\int_{\partial D}\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|^2,$$

$$\int_D|\lambda|^{1/2}\,|\mathbf{u}|^2\leqslant C\int_{\partial D}|\mathbf{u}|^2.$$

Hence, by the Schwartz inequality,

$$\int_D\left|\frac{\partial\mathbf{u}}{\partial x_n}\right|\,|\lambda|\,|\mathbf{u}|\leqslant C\left\|\frac{\partial\mathbf{u}}{\partial x_n}\right\|_{L^2(\partial D)}\| |\lambda|^{1/2}\mathbf{u}\|_{L^2(\partial D)}$$

This, together with (1.20), implies the lemma.

Remark 1.20. If $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{f}$ in D and $\mathbf{u}$ and its derivatives are suitably small at ∞, then a similar argument to that in the proof of Lemma 1.18 yields

$$\|\nabla\mathbf{u}\|_{L^2(\partial D)}+\| |\lambda|^{1/2}\mathbf{u}\|_{L^2(\partial D)}\leqslant C\left\|\frac{\partial\mathbf{u}}{\partial x_n}\right\|_{L^2(\partial D)}+\frac{C}{|\lambda|^{1/4}}\|\mathbf{f}\|_{L^2(D)}.$$

Moreover, if $\operatorname{supp}\mathbf{f}\subset B(x_0, r_0)$ for some $x_0\in\mathbb{R}^n$, $r_0>0$, then

$$\|\nabla\mathbf{u}\|_{L^2(\partial D)}+\| |\lambda|^{1/2}\mathbf{u}\|_{L^2(\partial D)}\leqslant C\left\|\frac{\partial\mathbf{u}}{\partial x_n}\right\|_{L^2(\partial D)}+C\|\mathbf{f}\|_{L^2(D)}.$$

In this case, C also depends on r_0.

2. The Dirichlet Problem

This section is devoted to the proof of Therems 0.10 and 0.12 in the Introduction.

We start with a Caccioppoli inequality.

Lemma 2.1. *Let* $D(R)=B(X_0, R)\cap\Omega$ *for some* $X_0\in\bar{\Omega}$ *and* $R>0$. *Suppose* $\mathbf{u}\in H^1(D(4R))$ *and*

$$\begin{aligned} -A\mathbf{u}+\lambda\mathbf{u}&=\mathbf{0} &&\text{in } D(3R)\\ \mathbf{u}&=\mathbf{0} &&\text{on } \partial\Omega\cap\partial D(3R). \end{aligned}$$

Then, for any integer $k>0$,

$$\int_{D(R)}|\nabla\mathbf{u}|^2+|\lambda|\int_{D(R)}|\mathbf{u}|^2\leqslant\frac{C_k}{(1+|\lambda|\,R^2)^k R^2}\int_{D(2R)}|\mathbf{u}|^2.$$

Proof. Let $\eta\in C_0^\infty(\mathbb{R}^n)$ such that $\eta\equiv 1$ on $B(X_0, R)$, $\eta\equiv 0$ outside $B(X_0, 2R)$ and $|\nabla\eta|\leqslant C/R$. It follows from integration by parts and the assumption $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0}$ in $D(3R)$ and $\mathbf{u}=\mathbf{0}$ on $\partial\Omega\cap\partial D(3R)$ that

$$\int_{D(2R)}a_{ij}^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial}{\partial x_j}(\bar{u}^\beta\eta^2)+\lambda\int_{D(2R)}|\mathbf{u}|^2\eta^2=0.$$

Thus,

$$\begin{aligned}&\left|\int_{D(2R)}a_{ij}^{\alpha\beta}\frac{\partial}{\partial x_i}(u^\alpha\eta)\frac{\partial}{\partial x_j}(\bar{u}^\beta\eta)+\lambda\int_{D(2R)}|\mathbf{u}\eta|^2\right|\\ &\qquad\leqslant\frac{C}{R}\int_{D(2R)}|\mathbf{u}|\,|\nabla(\mathbf{u}\eta)|+\frac{C}{R^2}\int_{D(2R)}|\mathbf{u}|^2.\end{aligned}$$

By means of Fourier transform

$$c\int_{D(2R)} |\nabla(\mathbf{u}\eta)|^2 \leqslant \int_{D(2R)} a_{ij}^{\alpha\beta} \frac{\partial}{\partial x_i}(u^\alpha \eta)\cdot \frac{\partial}{\partial x_j}(\bar{u}^\beta \eta).$$

Since $\lambda \in \Sigma_\theta$, by the usual Caccioppoli argument, we obtain

$$\int_{D(R)} |\nabla \mathbf{u}|^2 + |\lambda| \int_{D(R)} |\mathbf{u}|^2 \leqslant \frac{C}{R^2}\int_{D(2R)} |\mathbf{u}|^2.$$

The lemma then follows by repeating above procedure k times.

Using Lemma 2.1 and a standard argument, we see that, if $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0}$ in $B(X, 2R)$, then

$$|\nabla^\gamma u(X)|^2 \leqslant \frac{C_{k,\gamma}}{(1+|\lambda|\, R^2)^k\, R^{n+|\gamma|}} \int_{B(X,R)} |\mathbf{u}(Y)|^2\, dY \tag{2.2}$$

where $\gamma = (\gamma_1, ..., \gamma_n)$ is a multi index.

Now, let $\Gamma_\lambda(X, Y) = \Gamma_\lambda(X-Y)$ denote the matrix fundamental solution of $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0}$ in $\mathbb{R}^n$ with pole at X. It follows from (2.2) that

$$|\nabla^\gamma \Gamma_\lambda(X, Y)| \leqslant \frac{C_{k,\gamma}}{(1+|\lambda|\, |X-Y|^2)^k\, |X-Y|^{n-2+|\gamma|}}. \tag{2.3}$$

Also, note that,

$$-A(\Gamma_\lambda(X)-\Gamma_0(X)) + \lambda(\Gamma_\lambda(X)-\Gamma_0(X)) = -\lambda \Gamma_0(X).$$

Thus, we may write

$$\Gamma_\lambda(X)-\Gamma_0(X) = -\lambda \int_{\mathbb{R}^n} \Gamma_\lambda(X-Z)\, \Gamma_0(Z)\, dZ.$$

From this and (2.3), a computation shows that, for $|X| < 2/\sqrt{|\lambda|}$,

$$\begin{aligned} |\nabla(\Gamma_\lambda(X)-\Gamma_0(X))| &\leqslant \frac{C\,|\lambda|}{|X|^{n-3}} && \text{if } n>3,\\ |\nabla(\Gamma_\lambda(X)-\Gamma_0(X))| &\leqslant C\,|\lambda| \log \frac{C}{|X|\,\sqrt{|\lambda|}} && \text{if } n=3. \end{aligned} \tag{2.4}$$

For $\mathbf{g} \in L^p(\partial\Omega)$, $1<p<\infty$, we define the single layer potential

$$\mathcal{S}_\lambda(\mathbf{g})(X) = \int_{\partial\Omega} \Gamma_\lambda(X-Q)\, \mathbf{g}(Q)\, dQ \qquad \text{for } X \in \mathbb{R}^n \backslash \partial\Omega. \tag{2.5}$$

Using estimate (2.3) and (2.4), it is not very hard to show that

$$
\begin{aligned}
&(\nabla \mathcal{S}_\lambda(\mathbf{g}))^*(P)\\
&\leqslant C \sup_{\varepsilon>0} \left| \int_{|P-Q|>\varepsilon,\, Q\in\partial\Omega} \nabla_P \Gamma_0(P-Q)\, \mathbf{g}(Q)\, dQ \right| + CM_{\partial\Omega}(\mathbf{g})(P) \qquad (2.6)
\end{aligned}
$$

for $P\in\partial\Omega$, where $M_{\partial\Omega}(\mathbf{g})$ denotes the Hardy–Littlewood maximal function of $\mathbf{g}$ on $\partial\Omega$. It then follows by well-known techniques from the theorem of Coifman, McIntosh and Meyer [4] that, $\nabla\mathcal{S}_\lambda(\mathbf{g})$ has nontangential limits a.e. on $\partial\Omega$ and for $1<p<\infty$,

$$
\|(\nabla \mathcal{S}_\lambda(\mathbf{g}))^*\|_{L^p(\partial\Omega)} + \|(|\lambda|^{1/2} \mathcal{S}_\lambda(\mathbf{g}))^*\|_{L^p(\partial\Omega)} \leqslant C\, \|\mathbf{g}\|_{L^p(\partial\Omega)} \qquad (2.7)
$$

with constant C independent of λ.

We are now in a position to give

Proof of Theorem 0.10. We will only give the proof of the theorem for the case $p=2$. The extension to $2-\delta<p<2+\delta$ for some $\delta>0$ can be accomplished by a real variable argument of Dahlberg and Kenig [6, 11].

For the case $p=2$, the uniqueness follows easily from Lemma 1.3, while the existence of the solution follows from the argument in [9]. We will concentrate on the estimate

$$
\|(\mathbf{u})^*\|_{L^2(\partial\Omega)} \leqslant C\, \|\mathbf{g}\|_{L^2(\partial\Omega)}. \qquad (2.8)
$$

To this end, we fix a coordinate cylinder $Z(Q_0, r_0)$ for $\partial\Omega$. Without loss of generality, we may assume that

$$
\begin{aligned}
10Z\cap\Omega &= \{(X', x_n)\in\mathbb{R}^n,\ x_n>\varphi(X')\}\cap\Omega,\\
10Z\cap\partial\Omega &= \{(X', x_n)\in\mathbb{R}^n,\ x_n=\varphi(X')\}\cap\partial\Omega
\end{aligned}
$$

where φ is a Lipschitz function an $10Z$ is the dilation of Z about $Q_0=(0,\varphi(0))\in\partial\Omega$.

Let $\eta\in C_0^\infty(\mathbb{R}^n)$ such that $\eta\equiv 1$ on $3Z$ and $\eta\equiv 0$ outside $4Z$. By the Green's formula, we may write for $X\in 2Z\cap\Omega$,

$$
\begin{aligned}
\mathbf{u}(X) &= \int_{\partial D} \frac{\partial\Gamma_\lambda}{\partial\nu_Q}(X-Q)\, \mathbf{u}(Q)\, \eta(Q)\, dQ - \int_{\partial D} \Gamma_\lambda(X-Q) \frac{\partial}{\partial\nu_Q}(\mathbf{u}\eta)\, dQ\\
&\quad + \int_D \Gamma(X-Y)(-A+\lambda)(\mathbf{u}\eta)(Y)\, dY\\
&= \mathbf{u}_1(X) + \mathbf{u}_2(X) + \mathbf{u}_3(X)
\end{aligned}
$$

where $D=\{(X', x_n)\in\mathbb{R}^n : x_n>\varphi(X')\}$ and $\partial/\partial\nu_Q$ denotes the conormal derivative (see (1.5)).

First, it follows from (2.7) that

$$\|(\mathbf{u}_1)^*\|_{L^2(Z\cap\partial\Omega)} \leqslant C\,\|\mathbf{g}\|_{L^2(\partial\Omega)}.$$

Next, note that, by the system $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0}$ and integration by parts, for $X\in 2Z\cap\Omega$,

$$\begin{aligned}|\mathbf{u}_3(X)| &\leqslant C\int_\Omega |\nabla\Gamma_\lambda(X-Y)|\,|\mathbf{u}(Y)|\,|\nabla\eta(Y)|\,dY\\ &\quad + C\int_\Omega |\Gamma_\lambda(X-Y)|\,|\mathbf{u}(Y)|\,|\nabla^2\eta(Y)|\,dY\\ &\quad + C\int_{\partial\Omega} |\Gamma_\lambda(X-Q)|\,|\mathbf{u}(Q)|\,|\nabla\eta(Q)|\,dQ\\ &\leqslant C\int_\Omega |\mathbf{u}(Y)|\,dY + C\int_{\partial\Omega}|\mathbf{g}(Q)|\,dQ,\end{aligned}$$

where in the second inequality we have used the fact $|\nabla\eta|=0$ on $3Z$. Clearly, this implies that

$$\|(\mathbf{u}_3)^*\|_{L^2(Z\cap\partial\Omega)} \leqslant C\,\|\mathbf{u}\|_{L^2(\Omega)} + C\,\|\mathbf{g}\|_{L^2(\partial\Omega)} \leqslant C\,\|\mathbf{g}\|_{L^2(\partial\Omega)},$$

where the second inequality follows from Lemma 1.3.

Finally, we need to show

$$\|(\mathbf{u}_2)^*\|_{L^2(Z\cap\partial\Omega)} \leqslant C\,\|\mathbf{g}\|_{L^2(\partial\Omega)}. \tag{2.9}$$

Let $\psi\in C_0^\infty(\mathbb{R}^n)$ such that $\psi\equiv 1$ on $5Z$ and $\psi\equiv 0$ outside $6Z$. We define a function $\mathbf{v}$ on D by

$$\mathbf{v}(X', x_n) = -\int_{x_n}^\infty (\mathbf{u}\psi)(X', t)\,dt. \tag{2.10}$$

Clearly,

$$\frac{\partial\mathbf{v}}{\partial x_n} = \mathbf{u}\psi \qquad \text{in } D \tag{2.11}$$

and

$$-A\mathbf{v}+\lambda\mathbf{v} = -\int_{x_n}^\infty (-A+\lambda)(\mathbf{u}\psi)(X', t)\,dt = \mathbf{f}. \tag{2.12}$$

With the system $-A\mathbf{u}+\lambda\mathbf{u}=\mathbf{0}$ in Ω, it is easy to see that

$$|\mathbf{f}(X', x_n)| \leqslant C \int_{x_n}^{\infty} |\mathbf{u}\,\nabla^2\psi|\,dt + C \int_{x_n}^{\infty} |\nabla\mathbf{u}\,\nabla\psi|\,dt. \tag{2.13}$$

Now, using (2.11), we may write for $X \in 2Z \cap \Omega$

$$\begin{aligned} u_2^{\alpha}(X) &= \int_{\partial D} \Gamma_\lambda^{\alpha\beta}(X-Q)\, a_{ij}^{\beta s} N_i \left[\frac{\partial}{\partial Q_j}(u^s\eta)\right] \psi\, dQ \\ &= \int_{\partial D} \Gamma_\lambda^{\alpha\beta}(X-Q)\, a_{ij}^{\beta s} N_i \left[\frac{\partial}{\partial Q_j}(u^s\psi)\right] \eta\, dQ + E \\ &= \int_{\partial D} \Gamma_\lambda^{\alpha\beta}(X-Q)\, a_{ij}^{\beta s} N_i \frac{\partial^2 v^s}{\partial Q_j\, Q_n} \eta\, dQ + E \end{aligned}$$

where E contains two terms which can be handled easily. By (2.12), the first term above equals

$$\begin{aligned} &\int_{\partial D} \Gamma_\lambda^{\alpha\beta}(X-Q) \left[a_{ij}^{\beta s} N_i \frac{\partial}{\partial Q_n} - a_{ij}^{\beta s} N_n \frac{\partial}{\partial Q_i}\right] \frac{\partial v^s}{\partial Q_j} \cdot \eta\, dQ \\ &\quad + \int_{\partial D} \Gamma_\lambda^{\alpha\beta}(X-Q)\, \lambda v^\beta \eta\, dQ \\ &\quad - \int_{\partial D} \Gamma_\lambda^{\alpha\beta}(X-Q)\, f^\beta(Q)\, \eta(Q)\, dQ = I_1 + I_2 + I_3. \end{aligned}$$

By (2.13) and interior estimates, it is easy to see that, if $Q \in \operatorname{supp} \eta \cap \partial D$,

$$|\mathbf{f}(Q)| \leqslant C \int_{\Omega} |\mathbf{u}|.$$

It follows that

$$\|(I_3)^*\|_{L^2(Z\cap\partial\Omega)} \leqslant C\, \|\mathbf{u}\|_{L^2(\Omega)} \leqslant C\, \|\mathbf{g}\|_{L^2(\partial\Omega)}.$$

For I_1, note that, for β, s, j fixed,

$$a_{ij}^{\beta s} N_i \frac{\partial}{\partial Q_n} - a_{ij}^{\beta s} N_n \frac{\partial}{\partial Q_i}$$

is a tangential derivative operator on $\partial\Omega$ which can be passed on to the factor $\Gamma_\lambda^{\alpha\beta}(X-Q)$ in I_1. Thus, it follows from (2.7) and Remark 1.20 that

$$\begin{aligned}
&\|(I_1)^*\|_{L^2(Z\cap\partial\Omega)} + \|(I_2)^*\|_{L^2(Z\cap\partial\Omega)} \\
&\quad \leqslant C\,\|\nabla \mathbf{v}\|_{L^2(\partial D)} + C\,\|\,|\lambda|^{1/2}\mathbf{v}\|_{L^2(\partial D)} \\
&\quad \leqslant C\left\|\frac{\partial \mathbf{v}}{\partial x_n}\right\|_{L^2(\partial D)} + C\,\|\mathbf{f}\|_{L^2(D)} \leqslant C\,\|\mathbf{g}\|_{L^2(\partial\Omega)} + C\,\|\mathbf{f}\|_{L^2(D)}.
\end{aligned}$$

To finish the proof, notice that, by (2.13) and the Hardy inequality (see [19, p. 272])

$$\|\mathbf{f}\|_{L^2(D)} \leqslant C\,\|d\mathbf{u}\|_{L^2(\Omega)} + C\,\|d\,\nabla\mathbf{u}\|_{L^2(\Omega)} \leqslant C\,\|\mathbf{u}\|_{L^2(\Omega)} \leqslant C\,\|\mathbf{g}\|_{L^2(\partial\Omega)}.$$

where $d=d(X)=\operatorname{dist}(X,\partial\Omega)$. Hence,

$$\|(I_1)^*\|_{L^2(Z\cap\partial\Omega)} + \|(I_2)^*\|_{L^2(Z\cap\partial\Omega)} \leqslant C\,\|\mathbf{g}\|_{L^2(\partial\Omega)}.$$

So (2.9) is proved, thus

$$\|(\mathbf{u})^*\|_{L^2(Z\cap\partial\Omega)} \leqslant C\,\|\mathbf{g}\|_{L^2(\partial\Omega)}.$$

The estimate (2.8) follows by covering $\partial\Omega$ with a finite collection of coordinate cylinders.

We close the section with

Proof of Theorem 0.12. The uniqueness follows from Lemma 1.3. For the existence, we will prove that the single layer potential $\mathcal{S}_\lambda : L^2(\partial\Omega)\to L_1^2(\partial\Omega)$ defined in (2.5) is invertible. Indeed, as in the case of $\lambda=0$ [9], by the jump relation of the conormal derivatives of $\mathcal{S}_\lambda$, Lemma 1.14, Remark 1.15 and an approximation argument, we have,

$$\|\mathbf{g}\|_\partial \leqslant C\{\|\nabla_{\tan}\mathcal{S}_\lambda(\mathbf{g})\|_\partial + \|\,|\lambda|^{1/2}\,\mathcal{S}_\lambda(\mathbf{g})\|_\partial + \|\mathcal{S}_\lambda(\mathbf{g})\|_\partial\} \tag{2.14}$$

where $\|\cdot\|_\partial$ denote the norm in $L^2(\partial\Omega)$.

Since $\mathcal{S}_0 : L^2(\partial\Omega)\to L_1^2(\partial\Omega)$ is invertible [9] and $\mathcal{S}_\lambda-\mathcal{S}_0 : L^2(\partial\Omega)\to L_1^2(\partial\Omega)$ is a compact operator by (2.4), we conclude that $\mathcal{S}_\lambda : L^2(\partial\Omega)\to L_1^2(\partial\Omega)$ is invertivle. Finally, it follows from (2.7) and (2.14) that the unique solution $\mathbf{u}$ with boundary data $\mathbf{g}$ satisfies

$$\|(\nabla\mathbf{u})^*\|_\partial \leqslant C\{\|\nabla_{\tan}\mathbf{g}\|_\partial + \|\,|\lambda|^{1/2}\,\mathbf{g}\|_\partial + \|\mathbf{g}\|_\partial\}.$$

The proof is finished.

3. Proof of Main Results

In this section we will give the proof of Theorems 0.3 and 0.5 in the Introduction.

Recall that $\Gamma_\lambda(X-Y)$ denotes the matrix of fundamental solutions in $\mathbb{R}^n$. Using Theorem 0.12, we may construct the matrix Green's function $G_\lambda(X, Y)$ in Ω. Indeed for $X \in \Omega$, let $v^X(Y)$ be the matrix valued solution to the Dirichlet problem (0.9) with boundary data

$$v^X(Q) = \Gamma_\lambda(X-Q) \qquad \text{on } \partial\Omega.$$

We have

$$G_\lambda(X, Y) = \Gamma_\lambda(X-Y) - v^X(Y). \tag{3.1}$$

By the interior estimate (2.2), if $|Y-X| \leqslant \frac{1}{2}\mathrm{dist}(X, \partial\Omega)$,

$$|G_\lambda(X, Y)| \leqslant \frac{C_k}{(1+|\lambda|\,|X-Y|^2)^k\,|X-Y|^{n-2}}, \qquad k>0. \tag{3.2}$$

Lemma 3.3. *Let $X \in \Omega$, $P \in \partial\Omega$ and $r = |X-P| \leqslant 2\,\mathrm{dist}(X, \partial\Omega)$. Then*

$$\int_{\partial\Omega\setminus\Delta(P,6r)} |(G_\lambda(X,\cdot))^*(Q)|^p\,dQ \leqslant C r^{(n-1)-(n-2)p}$$

where $\Delta(P, 5r) = \{Q \in \partial\Omega : |Q-P| \leqslant 5r\}$ and $|p-2| < \delta$, δ is the same as in Theorem 0.10.

Proof. We may assume that r is small.

Let $B(P, r)$ denote the ball in $\mathbb{R}^n$ of radius r centered at P. We apply Theorem 0.10 in the domain $\Omega\setminus B(P, 4r)$ to obtain

$$\begin{aligned}
&\int_{\partial\Omega\setminus\Delta(P,6r)} |G_\lambda(X,\cdot))^*(Q)|^p\,dQ \\
&\quad\leqslant C\int_{\Omega\cap\partial B(P,4r)} |G_\lambda(X,Q)|^p\,dQ \\
&\quad\leqslant C\int_{\Omega\cap\partial B(P,4r)} |\Gamma_\lambda(X-Q)|^p\,dQ + C\int_{\Omega\cap\partial B(P,4r)} |v^X(Q)|^p\,dQ
\end{aligned}$$

where we have used the fact $G_\lambda(X, Q) = 0$ on $\partial\Omega$. The first term can be handled easily by (2.3). To estimate the second term, let $q > p$ such that $((n-1)/(n-2)) < q < 2+\delta$. Then, by the Hölder inequality and Theorem 0.10

$$
\begin{aligned}
\int_{\Omega \cap \partial B(P, 4r)} & |v^X(Q)|^p \, dQ \\
& \leqslant C r^{(n-1)(1-p/q)} \left(\int_{\Omega \cap \partial B(P, 4r)} |v^X(Q)|^q \, dQ \right)^{p/q} \\
& \leqslant C r^{(n-1)(1-p/q)} \left(\int_{\partial\Omega} |\Gamma_\lambda(X-Q)|^q \, dQ \right)^{p/q} \\
& \leqslant C r^{(n-1)-(n-2)p}.
\end{aligned}
$$

LEMMA 3.4. *Let $X \in \Omega$ and $r = dist(X, \partial\Omega)$. Then*

$$
\int_{\Omega \setminus B(X, r/4)} |G_\lambda(X, Y)| \, dY \leqslant \frac{C}{1+|\lambda|} \qquad \text{if} \quad n=3 \tag{3.5}
$$

and

$$
\int_{\Omega \setminus B(X, r/4)} |G_\lambda(X, Y)| \, dY \leqslant \frac{C}{r^{(n-3)/2-\varepsilon}(1+|\lambda|)^{(n+1)/4-(\varepsilon/2)}} \qquad \text{if} \quad n \geqslant 3, \tag{3.6}
$$

where $\varepsilon > 0$ depends on Ω and θ.

Proof. Let $P \in \partial\Omega$ such that $r = |X-P|$. For $R \geqslant r$, let $\Lambda(R) = \{Y \in \Omega, (R/4) \leqslant |Y-X| \leqslant 4R\}$ and $\tilde{\Lambda}(R) = \Lambda(R/2) \cup \Lambda(2R)$.

First, note that, by Lemma 2.1,

$$
\begin{aligned}
\int_{\Lambda(r)} & |G_\lambda(X, Y)| \, dY \\
& \leqslant C r^{n/2} \left(\int_{\Lambda(r)} |G_\lambda(X, Y)|^2 \, dY \right)^{1/2} \\
& \leqslant \frac{C_k r^{n/2}}{(1+|\lambda| r^2)^k} \left(\int_{\tilde{\Lambda}(r)} |G_\lambda(X, Y)|^2 \, dY \right)^{1/2} \\
& \leqslant \frac{C_k r^{n/2}}{(1+|\lambda| r^2)^k} \left\{ \left(\int_{\tilde{\Lambda}(r)} |\Gamma_\lambda(X-Y)|^2 \, dY \right)^{1/2} + \left(\int_{\tilde{\Lambda}(r)} |v^X(Y)|^2 \, dY \right)^{1/2} \right\} \\
& \leqslant \frac{C_k r^{n/2}}{(1+|\lambda| r^2)^k} \left\{ r^{-n/2+2} + r^{1/2} \left(\int_{\Delta(P, r)} |(v^X)^*|^2 (Q) \, dQ \right)^{1/2} \right\}
\end{aligned}
$$

$$\leqslant \frac{C_k r^{n/2}}{(1+|\lambda|\, r^2)^k}$$
$$\times \left\{ r^{-n/2+2} + r^{1/2} \cdot r^{(n-1)(1/2-1/q_0)} \left(\int_{\Delta(P,r)} |(v^X)^*|^{q_0} (Q)\, dQ \right)^{1/q_0} \right\}$$
$$\leqslant \frac{C_k r^{n/2}}{(1+|\lambda|\, r^2)^k}$$
$$\times \left\{ r^{-n/2+2} + r^{1/2} \cdot r^{(n-1)(1/2-1/q_0)} \left(\int_{\partial\Omega} |\Gamma_\lambda(X-Q)|^{q_0}\, dQ \right)^{1/q_0} \right\}$$
$$\leqslant \frac{C_k r^2}{(1+|\lambda|\, r^2)^k}$$

where $2 < q_0 < 2+\delta$ and we have used the Hölder inequality, Theorem 0.10 and (2.3).

Next, we claim that, for $R \geqslant 16r$,

$$\int_{A(R)} |G_\lambda(X, Y)|\, dY \leqslant C_k \left(\frac{r}{R}\right)^{2\varepsilon} \cdot \frac{R^{(n+1)/2}}{(1+|\lambda|\, R^2)^k\, r^{(n-3)/2}}, \qquad k>0 \quad (3.7)$$

for some $\varepsilon > 0$.

Assuming the claim for a moment, we give the proof of (3.5) and (3.6). If $|\lambda| \leqslant 1$, by (3.7),

$$\int_{A(R)} |G_\lambda(X, Y)|\, dY \leqslant C \left(\frac{r}{R}\right)^{\varepsilon} \cdot R^{(n+1)/2-\varepsilon} \cdot r^{-(n-3)/2+\varepsilon} \leqslant C \left(\frac{r}{R}\right)^{\varepsilon} \cdot r^{-(n-3)/2+\varepsilon}$$

where the boundedness of domain Ω is used. If $|\lambda| > 1$, let $k = ((n+1)/4) - (\varepsilon/2)$ in (3.7), we have

$$\int_{A(R)} |G_\lambda(X, Y)|\, dY \leqslant C \left(\frac{r}{R}\right)^{\varepsilon} \cdot |\lambda|^{(n+1)/4-\varepsilon/2} \cdot r^{-(n-3)/2+\varepsilon}.$$

Thus, for any $\lambda \in \Sigma_\theta$,

$$\int_{A(R)} |G_\lambda(X, Y)|\, dY \leqslant C \left(\frac{r}{R}\right)^{\varepsilon} \cdot \frac{1}{(1+|\lambda|)^{(n+1)/4-\varepsilon/2}\, r^{(n-3)/2-\varepsilon}}$$

(3.6) then follows from a summation argument.

To see (3.5), note that, by (3.7),

$$\int_{A(R)} |G_\lambda(X, Y)|\, dY \leqslant C \left(\frac{r}{R}\right)^{2\varepsilon} \cdot \frac{1}{1+|\lambda|}$$

for $n = 3$. (3.5) then follows.

Finally, we need to show (3.7).

To this end, we fix $q \in (2-\delta, 2)$. By the Hölder inequality, the Sobolev inequality and Lemma 2.1

$$\begin{aligned}
\int_{\Delta(R)} |G_\lambda(X, Y)|\, dY &\leqslant CR^{(3n+2)/4} R^{-n/2q} \left(\int_{\Delta(R)} |G_\lambda(X, Y)|^q\, dY \right)^{1/2q} \\
&\quad \times \left(\int_{\Delta(R)} |G_\lambda(X, Y)|^{2n/(n-2)}\, dY \right)^{(n-2)/4n} \\
&\leqslant CR^{(3n+2)/4} R^{-n/2q} \left(\int_{\Delta(R)} |G_\lambda(X, Y)|^q\, dY \right)^{1/2q} \\
&\quad \times \left(\int_{\Delta(R)} |\nabla_Y G_\lambda(X, Y)|^2\, dY \right)^{1/4} \\
&\leqslant \frac{CR^{3n/4} R^{-n/2q}}{(1+|\lambda|\, R^2)^k} \left(\int_{\Delta(R)} |G_\lambda(X, Y)|^q\, dY \right)^{1/2q} \\
&\quad \times \left(\int_{\bar{\Delta}(R)} |G_\lambda(X, Y)|^2\, dY \right)^{1/4}
\end{aligned}$$

It then follows from Lemma 3.3 that

$$\begin{aligned}
&\int_{\Delta(R)} |G_\lambda(X, Y)|\, dY \\
&\quad \leqslant \frac{C_k R^{(3n+1)/4 - (n-1)/2q}}{(1+|\lambda|\, R^2)^k} \left(\int_{\partial\Omega \setminus \Delta(P, 6r)} |(G(X, \cdot))^*(Q)|^q\, dq \right)^{1/2q} \\
&\quad \times \left(\int_{\partial\Omega \setminus \Delta(P, 6r)} |(G(X, \cdot))^*(Q)|^2\, dQ \right)^{1/4} \\
&\quad \leqslant C_k \left(\frac{r}{R} \right)^{2\varepsilon} \cdot \frac{R^{(n+1)/2}}{(1+|\lambda|\, R^2)^k\, r^{(n-3)/2}}
\end{aligned}$$

where $2\varepsilon = (n-1)\{1/(2q) - 1/4\} > 0$. (3.7) is then proved and the proof is complete.

Note that, by the estimate (3.2),

$$\begin{aligned}
&\int_{B(X, r/2)} |G_\lambda(X, Y)|\, dY \\
&\qquad \leqslant C \int_{B(X, r/2)} \frac{dY}{(1+|\lambda|\, |X-Y|^2)^2\, |X-Y|^{n-2}} \leqslant \frac{C}{1+|\lambda|},
\end{aligned}$$

where $r = \text{dist}(X, \partial\Omega)$. This, together with (3.5), implies that, if $n = 3$,

$$\int_\Omega |G_\lambda(X, Y)| \, dY \leqslant \frac{C}{1 + |\lambda|}. \tag{3.8}$$

We are now ready to give

Proof of Theorem 0.3. The case $n = 3$ follows from (3.8) and duality.

Assume $n \geqslant 4$. Let $\mathbf{f} \in L^p(\Omega)$, $p \geqslant 2$ and $\mathbf{u} \in H_0^1(\Omega)$ solve the system $-A\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}$ in Ω. We have

$$\begin{aligned}
\mathbf{u}(X) &= \int_\Omega G_\lambda(X, Y) \, \mathbf{f}(Y) \, dY \\
&= \int_{\Omega \setminus B(X, r/2)} G_\lambda(X, Y) \, \mathbf{f}(Y) \, dY + \int_{B(X, r/2)} G_\lambda(X, Y) \, \mathbf{f}(Y) \, dY \\
&= \mathbf{u}_1(X) + \mathbf{u}_2(X),
\end{aligned}$$

where $r = r(X) = \text{dist}(X, \partial\Omega)$. By (3.2),

$$|\mathbf{u}_2(X)| \leqslant C \int_\Omega \frac{|\mathbf{f}(Y)| \, dY}{(1 + |\lambda| \, |X - Y|^2)^2 \, |X - Y|^{n-2}}.$$

It follows that

$$\|\mathbf{u}_2\|_p \leqslant \frac{C}{1 + |\lambda|} \|\mathbf{f}\|_p \qquad \text{for} \quad 1 \leqslant p \leqslant \infty$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$.

We will show that there exists $\delta = \delta(\Omega) > 0$ such that

$$\|\mathbf{u}_1\|_{p_0} \leqslant \frac{C}{1 + |\lambda|} \|\mathbf{f}\|_{p_0} \qquad \text{for} \quad p_0 = \frac{2n}{n-3} + \delta. \tag{3.9}$$

By Lemma 3.4,

$$\|r^{(n-3)/2 - \varepsilon} \mathbf{u}_1\|_\infty \leqslant \frac{C}{(1 + |\lambda|)^{(n+1)/4 - \varepsilon/2}} \|\mathbf{f}\|_\infty. \tag{3.10}$$

We claim that

$$\|r^\alpha \mathbf{u}_1\|_2 \leqslant \frac{C}{(1 + |\lambda|)^\beta} \|\mathbf{f}\|_2 \tag{3.11}$$

for $-3/2 < \alpha \leqslant -1$ and $\beta = 1 + \alpha/2$.

(3.9) follows from (3.10) and (3.11). Indeed, by complex interpolation, we have

$$\|\mathbf{u}_1\|_{p_0} \leqslant \frac{C}{(1+|\lambda|)^{\gamma}} \|\mathbf{f}\|_{p_0}$$

where $(1/p_0) = t(1/2) + (1-t)\cdot(1/\infty)$, $0 = t\alpha + (1-t)((n-3)/2 - \varepsilon)$ and $\gamma = t\beta + (1-t)((n+1)/4 - \varepsilon/2)$. A computation shows

$$p_0 = \frac{2}{t}, \qquad t = \frac{\dfrac{n-3}{2} - \varepsilon}{\dfrac{n-3}{2} - \varepsilon - \alpha}, \qquad \gamma = 1.$$

We may choose α so close to $-3/2$ that $t < (n-3)/n$. Thus, $p_0 = 2/t > 2n/(n-3)$ and (3.9) is proved. Consequently,

$$\|\mathbf{u}\|_{p_0} \leqslant \frac{C}{(1+|\lambda|)} \|\mathbf{f}\|_{p_0}.$$

The theorem then follows by interpolation and duality.

It remains to show (3.11).

First, note that, by (3.2),

$$|\mathbf{u}_2(X)\, r^{\alpha}(X)| \leqslant C \int_{\Omega} \frac{|\mathbf{f}(Y)|\, dY}{(1+|\lambda|\,|X-Y|^2)^2\, |X-Y|^{n-2-\alpha}}.$$

Hence, (3.11) holds for $\mathbf{u}_2$ in the place of $\mathbf{u}_1$ for $-3/2 < \alpha \leqslant -1$, $\beta = 1 + \alpha/2$. Thus, it suffices to show

$$\|r^{\alpha}\mathbf{u}\|_2 \leqslant \frac{C}{(1+|\lambda|)^{\beta}} \|\mathbf{f}\|_2 \tag{3.12}$$

for $-3/2 < \alpha \leqslant -1$, $\beta = 1 + \alpha/2$.

By the Hardy inequality [19, p. 272], (3.12) follows from

$$\|r^{\alpha+1}\, \nabla \mathbf{u}\|_2 \leqslant \frac{C}{(1+|\lambda|)^{\beta}} \|\mathbf{f}\|_2. \tag{3.13}$$

To show (3.13), we let

$$\mathbf{v}(X) = \int_{\Omega} \Gamma_{\lambda}(X-Y)\, \mathbf{f}(Y)\, dY. \tag{3.14}$$

By (2.3),

$$|\nabla \mathbf{v}(X)| \leqslant C \int_{\Omega} \frac{|\mathbf{f}(Y)|\, dY}{(1+|\lambda|\,|X-Y|^2)^2\,|X-Y|^{n-1}}.$$

Hence,

$$\|\nabla \mathbf{v}\|_2 \leqslant \frac{C}{(1+|\lambda|)^{1/2}} \|\mathbf{f}\|_2. \tag{3.15}$$

Also, by the Calderón–Zygmund estimate.

$$\|\nabla^2 \mathbf{v}\|_2 \leqslant C\, \|\mathbf{f}\|_2. \tag{3.16}$$

Let $\Omega_t = \{X \in \Omega,\ r(X) \leqslant t\}$. It follows from the Hardy inequality, (3.15) and (3.16) that

$$\begin{aligned}
\|r^{\alpha+1} \nabla \mathbf{v}\|_{L^2(\Omega)} &\leqslant \|r^{\alpha+1} \nabla \mathbf{v}\|_{L^2(\Omega_t)} + \|r^{\alpha+1} \nabla \mathbf{v}\|_{L^2(\Omega \setminus \Omega_t)} \\
&\leqslant C\, \|r^{\alpha+2} \nabla^2 \mathbf{v}\|_{L^2(\Omega_{2t})} + \|r^{\alpha+1} \nabla \mathbf{v}\|_{L^2(\Omega\setminus\Omega_t)} \\
&\leqslant C t^{\alpha+2}\, \|\mathbf{f}\|_2 + C t^{\alpha+1}(1+|\lambda|)^{-1/2}\, \|\mathbf{f}\|_2 \\
&= C\{t^{\alpha+2} + t^{\alpha+1}(1+|\lambda|)^{-1/2}\}\, \|\mathbf{f}\|_2.
\end{aligned}$$

Here, the restriction $\alpha > -3/2$ is used in the second inequality since $\nabla \mathbf{v}$ may not vanish on $\partial\Omega$. Choosing $t = (1+|\lambda|)^{-1/2}$, we obtain

$$\|r^{\alpha+1} \nabla \mathbf{v}\|_2 \leqslant \frac{C}{(1+|\lambda|)^{1+\alpha/2}} \|\mathbf{f}\|_2 .$$

Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$. We still need to show

$$\|r^{\alpha+1} \nabla \mathbf{w}\| \leqslant \frac{C}{(1+|\lambda|)^{1+\alpha/2}} \|\mathbf{f}\|_2 \qquad \text{for} \quad -3/2 < \alpha \leqslant -1. \tag{3.17}$$

Clearly, $-A\mathbf{w} + \lambda \mathbf{w} = \mathbf{0}$ in Ω and $\mathbf{w} = -\mathbf{v}$ on $\partial\Omega$. It is not hard to see that

$$\|(\mathbf{w})^*\|_{\partial} \leqslant \|(\mathbf{u})^*\|_{\partial} + \|(\mathbf{v})^*\|_{\partial} \leqslant C\, \|\mathbf{u}\|_{H_0^1} + C\, \|\mathbf{f}\|_2 < +\infty.$$

Moreover, since $\mathbf{v}|_{\partial\Omega} \in L_1^2(\partial\Omega)$, we have by Theorem 0.12,

$$\begin{aligned}
\|(\nabla \mathbf{w})^*\|_{\partial} &\leqslant C\{\, \|\nabla_{\tan} \mathbf{w}\|_{\partial} + (1+|\lambda|)^{1/2}\, \|\mathbf{w}\|_{\partial}\} \\
&\leqslant C\{\, \|\nabla \mathbf{v}\|_{\partial} + (1+|\lambda|)^{1/2}\, \|\mathbf{v}\|_{\partial}\}.
\end{aligned}$$

Let $\mathbf{h} \in C_0^\infty(\mathbb{R}^n)$ such that $\langle \mathbf{h}, \mathbf{N} \rangle \geqslant c_0 > 0$ on $\partial\Omega$. Using integration by parts on the integral $\int_{\partial\Omega} \langle \mathbf{h}, \mathbf{N} \rangle \, |\nabla \mathbf{v}|^2 \, d\sigma$, we obtain

$$\begin{aligned}\|\nabla \mathbf{v}\|_\partial &\leqslant C\{\|\nabla \mathbf{v}\|_2 + \|\nabla^2 \mathbf{v}\|_2^{1/2} \|\nabla \mathbf{v}\|_2^{1/2}\} \\ &\leqslant C(1+|\lambda|)^{-1/4} \|\mathbf{f}\|_2.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\mathbf{v}\|_\partial &\leqslant C\{\|\mathbf{v}\|_2 + \|\nabla \mathbf{v}\|_2^{1/2} \|\mathbf{v}\|_2^{1/2}\} \\ &\leqslant C(1+|\lambda|)^{-3/4} \|\mathbf{f}\|_2.\end{aligned}$$

It follows that

$$\|(\nabla \mathbf{w})^*\|_\partial \leqslant C(1+|\lambda|)^{-1/4} \|\mathbf{f}\|_2. \tag{3.18}$$

Finally,

$$\begin{aligned}\|r^{\alpha+1} \nabla \mathbf{w}\|_{L^2(\Omega)} &\leqslant \|r^{\alpha+1} \nabla \mathbf{w}\|_{L^2(\Omega_t)} + \|r^{\alpha+1} \nabla \mathbf{w}\|_{L^2(\Omega \setminus \Omega_t)} \\ &\leqslant Ct^{\alpha+3/2} \|(\nabla \mathbf{w})^*\|_{L^2(\partial\Omega)} + Ct^{\alpha+1} \|\nabla \mathbf{w}\|_{L^2(\Omega)} \\ &\leqslant C\{t^{\alpha+3/2} + t^{\alpha+1}(1+|\lambda|)^{-1/4}\} \|\nabla \mathbf{w}\|_{L^2(\partial\Omega)}\end{aligned}$$

where we have used Lemma 1.3 in the last inequality. Set $t = (1+|\lambda|)^{-1/2}$. (3.17) then follows from (3.18).

The proof is complete.

We now turn to

Proof of Theorem 0.5. First, by (2.3) and a well known argument,

$$\begin{aligned}|\nabla \mathbf{u}(X)| &\leqslant C \int_{B(X,r/2)} \frac{|\mathbf{f}(Y)| \, dY}{(1+|\lambda| \, |X-Y|^2) \, |X-Y|^{n-1}} \\ &\quad + \frac{C}{r^{n+1}} \int_{B(X,r/2)} |\mathbf{u}(Y)| \, dY\end{aligned}$$

where $r = \operatorname{dist}(X, \partial\Omega)$. Hence,

$$|\nabla \mathbf{u}(X)| \leqslant \frac{C}{(1+|\lambda|)^{1/2}} M(\mathbf{f})(X) + CM(r^{-1} |\mathbf{u}|)(X)$$

where M is the Hardy–Littlewood maximal operator in $\mathbb{R}^n$. It then follows that

$$\|\nabla \mathbf{u}\|_p \leqslant \frac{C}{(1+|\lambda|)^{1/2}} \|\mathbf{f}\|_p + C \|r^{-1} \mathbf{u}\|_p \qquad \text{for} \quad 1 < p < \infty.$$

Thus, by interpolation, it suffices to show

$$\|r^{-1}\mathbf{u}\|_p \leqslant \frac{C}{(1+|\lambda|)^{1/2}} \|\mathbf{f}\|_p \tag{3.19}$$

for $p = p_1$ and its dual p_1' where $p_1 = (2n/(n-1)) + \varepsilon$, $\varepsilon > 0$.

Assume $n \geqslant 4$. We claim that (3.19) is a consequence of Theorem 0.3 and (3.12). Indeed, by complex interpolation,

$$\|r^{-1}\mathbf{u}\|_{p_1} \leqslant \frac{C}{(1+|\lambda|)^{1/2}} \|\mathbf{f}\|_{p_1}$$

where

$$\frac{1}{p_1} = \frac{t}{p_0} + \frac{1-t}{2}, \quad -1 = t \cdot 0 + (1-t)\alpha, \quad p_0 = \frac{2n}{n-3} + \delta.$$

A computation shows that we may choose α so close to $-3/2$ that $p_1 > 2n/(n-1)$. The case $p = p_1'$ follows in the same manner.

Finally, we consider the case $n = 3$. In this case, by (3.6),

$$\int_{\Omega \setminus B(X, r/4)} |G_\lambda(X, Y)| \, dY \leqslant \frac{Cr^\varepsilon}{(1+|\lambda|)^{1-\varepsilon/2}}, \qquad r = \operatorname{dist}(X, \partial\Omega), \qquad \varepsilon > 0.$$

Also, by (3.2),

$$\int_{B(X, r/2)} |G_\lambda(X, Y)| \, dY \leqslant \frac{Cr^\varepsilon}{(1+|\lambda|)^{1-\varepsilon/2}}.$$

It follows that

$$\int_\Omega |G_\lambda(X, Y)| \, dY \leqslant \frac{Cr^\varepsilon}{(1+|\lambda|)^{1-\varepsilon/2}}. \tag{3.20}$$

Hence, we have

$$\|r^{-\varepsilon}\mathbf{u}\|_\infty \leqslant \frac{C}{(1+|\lambda|)^{1-\varepsilon/2}} \|\mathbf{f}\|_\infty \qquad \text{for some} \quad \varepsilon > 0. \tag{3.21}$$

This, together with L^2-estimate (3.12), implies that

$$\|r^{-1}\mathbf{u}\|_{p_1} \leqslant \frac{C}{(1+|\lambda|)^{1/2}} \|\mathbf{f}\|_{p_1}, \qquad p_1 = 3 + \varepsilon, \quad \varepsilon > 0$$

by interpolation.

For the estimate in $L^{p'_1}$ space, we will show that

$$\int_\Omega |G_\lambda(X, Y)|\, r^{-\varepsilon}(Y)\, dY \leqslant \frac{C}{(1+|\lambda|)^{1-\varepsilon/2}} \qquad \text{for some} \quad \varepsilon > 0. \tag{3.22}$$

Clearly, this implies that

$$\|r^{-\varepsilon}\mathbf{u}\|_1 \leqslant \frac{C}{(1+|\lambda|)^{1-\varepsilon/2}} \|\mathbf{f}\|_1. \tag{3.23}$$

Again, the interpolation with (3.12) yields the desired estimate.

To show (3.22), we let $\mathcal{C} = \{ Y \in \Omega : |Y - X| < r(Y)/2\}$. Then, by (3.2), for $0 < \varepsilon < 2$,

$$\begin{aligned} \int_{\mathcal{C}} |G_\lambda(X, Y)|\, r^{-\varepsilon}(Y)\, dY &\leqslant C \int_\Omega \frac{dY}{(1+|\lambda|\, |X-Y|^2)^2\, |X-Y|^{n-2+\varepsilon}} \\ &\leqslant \frac{C}{(1+|\lambda|)^{1-\varepsilon/2}}. \end{aligned} \tag{3.24}$$

From the proof of (3.7), we have

$$\int_{\Lambda(R)} |G_\lambda(X, Y)|^2\, dY \leqslant C_k \cdot \left(\frac{r}{R}\right)^{2\varepsilon} \cdot \frac{R}{(1+|\lambda|\, R^2)^k}$$

where $\Lambda(R) = \{ Y \in \Omega : (R/4) \leqslant |Y - X| \leqslant 4R\}$, $R \geqslant r = r(X)$. It follows from the Schwartz inequality that

$$\begin{aligned} &\int_{\Lambda(R)\setminus\mathcal{C}} |G_\lambda(X, Y)|\, r^{-\varepsilon}(Y)\, dY \\ &\quad\leqslant \left(\int_{\Lambda(R)} |G_\lambda(X, Y)|^2\, dY\right)^{1/2} \left(\int_{\Lambda(R)\setminus\mathcal{C}} r^{-2\varepsilon}(Y)\, dY\right)^{1/2} \\ &\quad\leqslant C_k \left(\frac{r}{R}\right)^{\varepsilon} \cdot \frac{R^{2-\varepsilon}}{(1+|\lambda|\, R^2)^k} \\ &\quad\leqslant C \left(\frac{r}{R}\right)^{\varepsilon} \cdot \frac{1}{(1+|\lambda|)^{1-\varepsilon/2}}. \end{aligned}$$

Thus,

$$\int_{\Omega\setminus\mathcal{C}} |G_\lambda(X, Y)|\, r^{-\varepsilon}(Y)\, dY \leqslant \frac{C}{(1+|\lambda|)^{1-\varepsilon/2}}.$$

(3.22) is then proved and the proof of the theorem is finished.

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